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Stability of equilibria in a four-dimensional nonlinear model of a hydraulic servomechanism

A. HALANAY*, C.A. SAFTA[1], I. URSU[2] and F. URSU[2]
Department of Mathematics I, University Politehnica of Bucharest, Splaiul Independenţei 313, RO-060042, Bucharest, Romania (E-mail: halanay@mathem.pub.ro); [1]*Department of Hydraulic and Hydraulic Machinery, University Politehnica of Bucharest, Splaiul Independenţei 313, RO-060042, Bucharest, Romania (E-mail: saphta@hydrop.pub.ro);* [2]*'Elie Carafoli' National Institute for Aerospace Research, Bd. Iuliu Maniu 220, RO-061099, Bucharest, Romania (E-mail: iursu@aero.incas.ro);* **Author for correspondence; E-mail: halanay@mathem.pub.ro*

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Abstract. Starting from plysical laws a four-dimensional nonlinear model for mecano-hydraulic servomechanisms is deduced. The stability of its equilibria is analysed using a theorem of Lyapunov and Malkin to handle the critical case due to the presence of zero in the spectrum of the matrix of the linear part around equilibria. Stability diagrams are drawn and simulation results are presented through phase diagrams.

Key words: hydraulic servomechanism, Lyapunov stability, Routh–Hurwitz criterion

1. Introduction

Hydraulic servomechanisms are widely used in industries where heavy objects are manipulated or large forces and torques at high speeds are exerted. Features such as their large processing force and stiffness, good positioning and high payload capabilities, good power-to-weight ratio make this type of actuation systems appropriate for high-power industrial machinery such as positioning of aircraft control surfaces, position control of military gun turrets and antennas, material handling, construction, mining and agricultural equipment. The demanding performance specifications for such applications are high-bandwidth implying fast response time, high-accuracy and high-fidelity control. Such technical challenges have led researchers to examine how to improve hydraulic servomechanisms design and related synthesis. Among the numerous issues in the field we have to cite here contributions concerning: design of observers [1–3], feedback linearisation [4–6], feedback stabilisation [7], high-bandwidth control [8], robust control and neuro-fuzzy control [9, 10], absolute stable synthesis [11].

The sine-qua non of a functional system is stability; it concerns the basis of the system approach. Historically, stability of hydraulic servomechanisms has been studied using Taylor-type linearisation of flow equations [12–14] and harmonic balance method applied to special nonlinearities as saturation, dead band; see, for instance [15]. Generally speaking, the earliest method available in the study of linear systems was that of Routh and Hurwitz, where the stability of equilibrium points is ascertained by examining the coefficients of the characteristic equation associated to a linear time-invariant system, without having to find its roots. Then, works of Nyquist, Black and Bode in the 1930s laid the foundations for linear analysis in the frequency domain. But hardly about 30 years later the same progress takes place in nonlinear frequency analysis, one of the most important contribution being the discovery of the now famous criterion of Popov (see [16]).

Let us mention, in this context of hydraulic-servomechanism stability analysis, the investigation of critical cases, when stability is lost as some parameters vary. While in some cases Hopf bifurcations of equilibriua to limit cycles are present [17–19] there are also cases when the boundary of stability is safe [20].

An important problem for the analysis is finding a representative model of the physical system (what is meant by a 'representative model' is still open to debate in the field). This model is the result of a trade-off: to be complex enough to describe the physical behaviour of the system and simple enough not to compromise a qualitative analysis approach. Satisfying these rather contradictory requirements assumes the following steps: (a) derivation of a model that is as complex as possible in accordance with physical laws and designing constraints; (b) a certain adjustement of the developed model to allow the mathematical apparatus to provide system synthesis and analysis; (c) stability analysis providing design and optimisation rules in the theoretical framework; (d) validation of theoretical results by simulation and experiments (see also [21]).

To meet these goals, the paper is organised as follows. In Section 2, following [12] and [22], a mathematical model for a hydraulic servomechanism is introduced. In Section 3 the Lyapunov stability of equilibrium points is analysed. Diagrams for the stability regions in the space of two parameters and simulations for the solutions are presented in Section 4. A final section is devoted to concluding remarks.

2. A mathematical model for a hydraulic servomechanism

The foundations for modeling a hydraulic servomechanism were laid down in [23] and completed by important contributions in the following years [12–14].

A hydraulic servomechanism is in fact a valve-piston combination (Figure 1a). A thorough analysis of this combination to get dynamic performance is based on the pressure-flow equation of the hydraulic-control valve and the continuity equation (see [23]).

The valve considered in Figure 1b is an ideal 'two-land-four-way' [12] spool valve. An ideal valve is defined (see [15]) as one with the following properties:

a) the geometrical dimensions of the valve are symmetrical with respect to the axes X and Y;

b) the hydraulic conductances of the sleeve ports are the same for the same relative spool-sleeve displacements σ;

c) zero radial clearance δ_r and no overlap or underlap of the ports are postulated; also no inner pressure drops or losses in value occur;

d) if p_a is the supply pressure, we have $0 < p_i < p_a$, $i = 1, 2$ (see Figure 1b) (this means that the pressure in each chamber of the cylinder does not saturate or cavitate) and the flow through the valving orifices does not saturate.

An algebraic sign convention is chosen: positive for the incoming flow in the cylinder and negative for the outgoing flow. The σ-dependent port areas $a_i(\sigma)$, $i = 1, 2, 3, 4$ are also related to σ by a sign convention:

$$\sigma a_i(\sigma) > 0 \quad \forall \sigma \neq 0, \quad i = 1, 2, 3, 4 \tag{2.1}$$

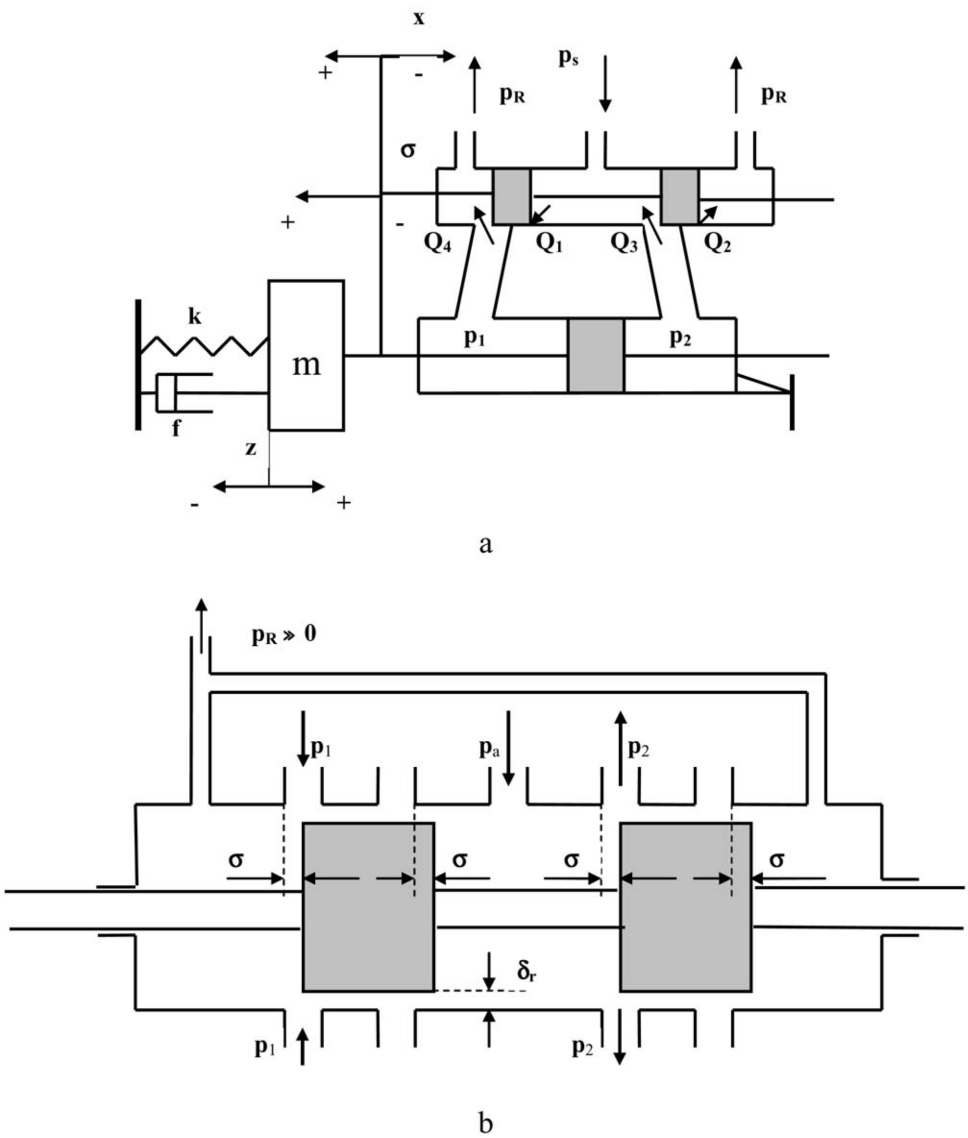


Figure 1. The model of a hydraulic servomechanism.

and $a(\sigma) = 0$ if and only if $\sigma = 0$. Let $G_i(\sigma)$, $i = 1, 2, 3, 4$, stand for the hydraulic conductances of the four ports

$$G_i(\sigma) = c_{d_i} a_i(\sigma) \sqrt{\frac{2}{\rho}}, \tag{2.2}$$

where c_{d_i} are the discharge coefficients and ρ is the mass density of the hydraulic fluid. Then the following equations for the flow through the valving ports can be written:

$$\begin{aligned} &Q_1 = G_1(\sigma)\sqrt{p_a - p_1}, \quad Q_2 = -G_2(\sigma)\sqrt{p_2}, \text{ if } \sigma > 0, \\ &Q_3 = -G_3(\sigma)\sqrt{p_a - p_2}, \quad Q_4 = G_4(\sigma)\sqrt{p_1}, \text{ if } \sigma < 0. \end{aligned} \tag{2.3}$$

By hypothesis (a) (indeed, great care is taken during manufacture to ensure that orifices are matched and symmetrical) we have

$$c_{d_1} = c_{d_2} = c_{d_3} = c_{d_4} := c_d, \; G_1(\sigma) = G_2(\sigma) = G_3(\sigma) = G_4(\sigma) := G(\sigma) \tag{2.4}$$

and, by (2.1), $\sigma G(\sigma) > 0 \; \forall \sigma \neq 0$.

The hydraulic-servomechanism model we consider was already introduced before in [12], [22], [7]. Because its development has not been described in full detail in the literature, we will present some details of the derivation of the equation of the valve-controlled piston. This equation, regarded in [24] as a nonholonomic constraint between the flow and the pressure, is based on the fact that the flow into and out of the cylinder is described by two components, one due to the movement of the piston and the other to compressibility effects. Denoting by Q_{in} and Q_{out} the volumetric flow rates into and out of the volume V, we may give the continuity equation

$$Q_{\text{in}} - Q_{\text{out}} = \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{B}\frac{\mathrm{d}p}{\mathrm{d}t}, \tag{2.5}$$

where p is the pressure inside V and B is the bulk modulus of oil.

Let S denote the effective area of the piston and V_0 the cylinder semi-volume. We first derive the equations assuming that the hypothesis on pressure saturation in (d) is satisfied. Using (2.1), (2.3), (2.4) and (2.5), we have that, for $\sigma > 0$, the following equations are satisfied:

$$\begin{cases} S\dot{z} + \dfrac{V_0 + Sz}{B}\dot{p}_1 = G(\sigma)\,\text{sgn}\,(p_a - p_1)\sqrt{|p_a - p_1|} \\ -S\dot{z} + \dfrac{V_0 - Sz}{B}\dot{p}_2 = -G(\sigma)\sqrt{|p_2|} \end{cases}. \tag{2.6}$$

Similarly, for $\sigma < 0$,

$$\begin{cases} S\dot{z} + \dfrac{V_0 + Sz}{B}\dot{p}_1 = G(\sigma)\sqrt{|p_1|} \\ -S\dot{z} + \dfrac{V_0 - Sz}{B}\dot{p}_2 = -G(\sigma)\,\text{sgn}\,(p_a - p_2)\sqrt{|p_a - p_2|} \end{cases}. \tag{2.7}$$

These two systems can be combined according to [22] as:

$$\begin{cases} S\dot{z} + \dfrac{V_0 + Sz}{B}\dot{p}_1 = \\ = |G(\sigma)|\,\mathrm{sgn}\,[p_a(1+\mathrm{sgn}\,\sigma) - 2p_1]\sqrt{\frac{|p_a(1+\mathrm{sgn}\,\sigma)-2p_1|}{2}} \\ -S\dot{z} + \dfrac{V_0 - Sz}{B}\dot{p}_2 = \\ = |G(\sigma)|\,\mathrm{sgn}\,[p_a(1-\mathrm{sgn}\,\sigma) - 2p_2]\sqrt{\frac{|p_a(1-\mathrm{sgn}\,\sigma)-2p_2|}{2}}. \end{cases} \tag{2.8}$$

The equations in (2.8) are very similar to the model used in [7]. In what follows we use these strongly nonlinear equations to model a hydraulic-servomechanism system that satisfies the hypothesis (d), stating that $0 < p_i < p_a$, $i = 1, 2$. These equations include conditions to avoid cavitation. Thus, for $\sigma > 0$, one gets

$$\begin{cases} S\dot{z} + \dfrac{V_0 + Sz}{B}\dot{p}_1 = c_d a(\sigma)\sqrt{\dfrac{2}{\rho}}\sqrt{p_a - p_1} \\ -S\dot{z} + \dfrac{V_0 - Sz}{B}\dot{p}_2 = -c_d a(\sigma)\sqrt{\dfrac{2}{\rho}}\sqrt{p_2} \end{cases} \tag{2.9}$$

and, for $\sigma < 0$,

$$\begin{cases} S\dot{z} + \dfrac{V_0 + Sz}{B}\dot{p}_1 = c_d a(\sigma)\sqrt{\dfrac{2}{\rho}}\sqrt{p_1} \\ -S\dot{z} + \dfrac{V_0 - Sz}{B}\dot{p}_2 = -c_d a(\sigma)\sqrt{\dfrac{2}{\rho}}\sqrt{p_a - p_2} \end{cases}. \tag{2.10}$$

The feedback linkage equation, generally taken as an algebraic linear equation connecting the input variable x, the output variable z and the 'error' σ, $\sigma = a_1 x + a_2 z$, will be herein given in the specific form (see [21, 25])

$$\sigma = \lambda(x - z), \tag{2.11}$$

with λ the coefficient of the feedback linkage.

In addition to Equations (2.9) or (2.10), we have to consider also the equation of motion of the piston assembly

$$m\ddot{z} + f\dot{z} + kz = S(p_1 - p_2), \tag{2.12}$$

where, in the case of a hydraulic servo actuating flight controls, m is the equivalent inertial load of the primary control surface reduced to the actuator road, f is an equivalent viscous-friction-force coefficient and k is the equivalent aerodynamic elastic-force coefficient. We will suppose also rectangular valve ports; thus denoting by w the valve-port width, we have (see [21, 26])

$$a(\sigma) = w\sigma. \tag{2.13}$$

3. Stability of equilibrium points

We investigate now the first-order systems of differential equations equivalent to (2.9), (2.11), (2.12) and (2.10), (2.11), (2.12), respectively. We treat the two cases separately.

For $\sigma > 0$, according to (2.9), (2.11), (2.12) and (2.13) we have the system

$$\begin{cases} \dot{z} = v \\ \dot{v} = -\dfrac{k}{m}z - \dfrac{f}{m}v + \dfrac{S}{m}p_1 - \dfrac{S}{m}p_2 \\ \dot{p}_1 = \dfrac{Bc_d w\lambda(x-z)}{V_0 + Sz}\sqrt{\dfrac{2}{\rho}(p_a - p_1)} - \dfrac{BSv}{V_0 + Sz} \\ \dot{p}_2 = -\dfrac{Bc_d w\lambda(x-z)}{V_0 - Sz}\sqrt{\dfrac{2}{\rho}p_2} + \dfrac{BSv}{V_0 - Sz}. \end{cases} \tag{3.1}$$

An easy computation leads to the equilibrium points

$$z = \tilde{z} = x,\ \ v = \tilde{v} = 0,\ \ p_1 = \tilde{p}_1 = \tilde{p}_2 + \frac{k}{S}x,\ \ p_2 = \tilde{p}_2 \tag{3.2}$$

with $0 < \tilde{p}_2 < p_a$, $0 < \tilde{p}_2 + \dfrac{k}{S}x < p_a$ and $|x| < \dfrac{V_0}{S}$.

Next we perform a translation to zero by

$$y_1(t) = z(t) - x,\ y_2(t) = v(t),\ y_3(t) = p_1(t) - \tilde{p}_2 - \frac{k}{S}x,\ y_4(t) = p_2(t) - \tilde{p}_2 \tag{3.3}$$

and denote for convenience $C = Bc_d w\lambda\sqrt{\dfrac{2}{\rho}}$.

We need to investigate the stability of the zero solution for the following system of equations

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\dfrac{k}{m}y_1 - \dfrac{f}{m}y_2 + \dfrac{S}{m}y_3 - \dfrac{S}{m}y_4 \\ \dot{y}_3 = -\dfrac{Cy_1}{V_0 + Sx + Sy_1}\sqrt{p_a - \dfrac{k}{S}x - \tilde{p}_2 - y_3} - \dfrac{BSy_2}{V_0 + Sx + Sy_1} \\ \dot{y}_4 = \dfrac{Cy_1}{V_0 - Sx - Sy_1}\sqrt{\tilde{p}_2 + y_4} + \dfrac{BSy_2}{V_0 - Sx - Sy_1}. \end{cases} \tag{3.4}$$

The Jacobian matrix of the terms on the right in (3.4), calculated in $(y_1, y_2, y_3, y_4) = \mathbf{0}$, is

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k}{m} & -\dfrac{f}{m} & \dfrac{S}{m} & -\dfrac{S}{m} \\ -\dfrac{C}{V_0 + Sx}\sqrt{p_a - \tilde{p}_2 - \dfrac{k}{S}x} & -\dfrac{BS}{V_0 + Sx} & 0 & 0 \\ \dfrac{C\sqrt{\tilde{p}_2}}{V_0 - Sx} & \dfrac{BS}{V_0 - Sx} & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Its characteristic polynomial is $Q_1(\lambda) = \lambda P_1(\lambda)$ with

$$P_1(\lambda) = \lambda^3 + \frac{f}{m}\lambda^2 + \frac{1}{m}\left(\frac{2BS^2V_0}{V_0^2 - S^2x^2} + k\right)\lambda + \frac{SC}{m}\left(\frac{\sqrt{\tilde{p}_2}}{V_0 - Sx} + \frac{\sqrt{p_a - \tilde{p}_2 - \frac{k}{S}x}}{V_0 + Sx}\right).$$

The coefficients of P_1 are positive so, according to the well-known Routh-Hurwitz criterion, P_1 is a stable polynomial if and only if

$$\frac{f}{m}\left(\frac{2BS^2V_0}{V_0^2 - S^2x^2} + k\right) - SC\left(\frac{\sqrt{\tilde{p}_2}}{V_0 - Sx} + \frac{\sqrt{p_a - \tilde{p}_2 - \frac{k}{S}x}}{V_0 + Sx}\right) > 0. \tag{3.6}$$

We assume that (3.6) is satisfied and continue the study of the Lyapunov stability of the zero solution of (3.4) following [27, Chapter IV, Sections 31–34]. First, we perform a transformation to modify (3.4) into a system having an equation having zero linear part (with respect to zero). We consider the linear system

$$\dot{\vec{x}} = A_1\vec{x} \tag{3.7}$$

with A_1 given in (3.5) and $\vec{x} = (x_1, x_2, x_3, x_4)^T$, and introduce

$$\eta = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4, \tag{3.8}$$

such that $\dot{\eta} = 0$. A straightforward identification gives

$$\begin{aligned} a_1 &= \frac{BS}{V_0 - Sx}\left(\frac{\sqrt{\tilde{p}_2}}{\sqrt{p_a - \tilde{p}_2 - \frac{k}{S}x}} - 1\right), \quad a_2 = 0, \\ a_3 &= \frac{V_0 + Sx}{V_0 - Sx}\frac{\sqrt{\tilde{p}_2}}{\sqrt{p_a - \tilde{p}_2 - \frac{k}{S}x}}, \quad a_4 = 1. \end{aligned} \tag{3.9}$$

If now we put $y_4 = y - a_1y_1 - a_3y_3$, Equations (3.4) become

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \left(\frac{S}{m}a_1 - \frac{k}{m}\right)y_1 - \frac{f}{m}y_2 + \frac{S}{m}(1 + a_3)y_3 - \frac{S}{m}y \\ \dot{y}_3 = -\frac{Cy_1}{V_0 + Sx + Sy_1}\sqrt{p_a - \tilde{p}_2 - \frac{k}{S}x - y_3} - \frac{BSy_2}{V_0 + Sx + Sy_1} \\ \dot{y} = \frac{Cy_1}{V_0 - Sx - Sy_1}\sqrt{\tilde{p}_2 + y - a_1y_1 - a_3y_3} + \\ \quad +y_2\left(\frac{BS}{V_0 - Sx - Sy_1} + a_1 - a_3\frac{BS}{V_0 + Sx + Sy_1}\right) - \\ \quad -\frac{a_3Cy_1}{V_0 + Sx + Sy_1}\sqrt{p_a - \tilde{p}_2 - \frac{k}{S}x - y_3}. \end{cases} \tag{3.10}$$

Introducing $\xi_3 = y_3 - \dfrac{y}{1+a_3}$, we eliminate y from the linear part of the first three equations and finally transform (3.4) into

$$\begin{cases}
\dot{y}_1 = y_2 \\
\dot{y}_2 = \left(\dfrac{S}{m}a_1 - \dfrac{k}{m}\right) y_1 - \dfrac{f}{m} y_2 + \dfrac{S}{m}(1+a_3)\xi_3 \\
\dot{\xi}_3 = -\dfrac{Cy_1}{V_0+Sx+Sy_1}\sqrt{p_a - \tilde{p}_2 - \dfrac{k}{S}x - \xi_3 - \dfrac{y}{1+a_3}} - \dfrac{BSy_2}{V_0+Sx+Sy_1} - \\
-\dfrac{1}{1+a_3}\left[\dfrac{Cy_1}{V_0-Sx-Sy_1}\sqrt{\tilde{p}_2 - a_1y_1 - a_3\xi_3 + \dfrac{1-a_3}{1+a_3}y} + \right. \\
+y_2\left(\dfrac{BS}{V_0-Sx-Sy_1} + a_1 - \dfrac{BSa_3}{V_0+Sx+Sy_1}\right) - \\
\left. -\dfrac{Ca_3y_1}{V_0+Sx+Sy_1}\sqrt{p_a - \tilde{p}_2 - \dfrac{k}{S}x - \xi_3 - \dfrac{y}{1+a_3}}\right] \\
\dot{y} = \dfrac{Cy_1}{V_0-Sx-Sy_1}\sqrt{\tilde{p}_2 - a_1y_1 - a_3\xi_3 + \dfrac{1-a_3}{1+a_3}y} + \\
\quad +y_2\left(\dfrac{BS}{V_0-Sx-Sy_1} + a_1 - \dfrac{BSa_3}{V_0+Sx+Sy_1}\right) - \\
\quad -\dfrac{a_3Cy_1}{V_0+Sx+Sy_1}\sqrt{p_a - \tilde{p}_2 - \dfrac{k}{S}x - \xi_3 - \dfrac{1}{1+a_3}y}.
\end{cases} \tag{3.11}$$

Recall now the definition of Lyapunov stability that we investigate: the zero solution of an autonomous system $x' = f(x)$ is Lyapunov-stable if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, if $|x_0| < \delta_\varepsilon$, the (maximal) solution of the Cauchy problem $x' = f(x)$, $x(0) = x_0$ is defined for all $t \geq 0$ and satisfies $|x(t)| < \varepsilon\ \forall t > 0$.

As already mentioned, we use the results from [27, Chapter IV, Section 34], namely the Theorem of Lyapunov (see [28]) in the general form proved by Malkin; we state it only for the autonomous case.

Theorem. *Given the autonomous system of differential equations*

$$\dot{x} = Ax + f(x,y), \quad \dot{y} = \mathbf{g}(x,y), \tag{3.12}$$

where A is a real $n \times n$ matrix, f: $G_1 \times G_2 \to \mathbf{R}^n$, g: $G_1 \times G_2 \to \mathbf{R}^m$ are (real) analytic functions, G_1, G_2 are open neighbourhoods of the origin in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Suppose that $\sigma(A) \subset \{z \in \mathbf{C} | \mathfrak{Re}\, z < 0\}$ (so $\mathbf{A}$ is stable), that $f(0,y) = g(0,y) = 0$, $(\forall) y \in G_2$ and that $f'(0,0) = g'(0,0) = \mathbf{0}$, so the Taylor series of f and g contain no power less than two.

Then the zero solution of (3.12) is Lyapunov stable. Even more, if $|x(0)|, |y(0)|$ are small enough, we have $\lim_{t\to\infty} x_j(t) = 0$, $j = 1, \ldots, n$ and $\lim_{t\to\infty} y_k(t) = \alpha_k$, $k = 1, \ldots, m$ with $(\alpha_1, \ldots, \alpha_m) \in G_2$.

We go back to the system (3.11). Here $n = 3$, $m = 1$. If f_3 and f_4 denote the terms on the right in Equations 3 and 4 of the system (3.11), we have

$$f_3(y_1, y_2, \xi_3, y) = \frac{\partial f_3}{\partial y_1}(\mathbf{0})y_1 + \frac{\partial f_3}{\partial y_2}(\mathbf{0})y_2 + \tilde{f}_3(y_1, y_2, \xi_3, y)$$

with

$$\tilde{f}_3(0, 0, 0, y) = f_3(0, 0, 0, y) = f_4(0, 0, 0, y) = 0 \tag{3.13}$$

for every y in a neighbourhood of zero. Note also that f_4 has first partial derivatives vanishing at $(0, 0, 0, 0)$.

Applying the Theorem of Lyapunov-Malkin stated above, we conclude that the zero solution of (3.11) is stable by Lyapunov and that every solution with small enough initial conditions has a limit for $t \to \infty$ in the form $(0, 0, 0, \alpha)$. Here $\alpha = y(0) + \int_0^\infty f_4\,[y_1(t), y_2(t), \xi_3(t), y(t)]\,\mathrm{d}t$ and, as results from the proof of the Theorem [27, pp. 115–116], the decay to zero of y_1, y_2, ξ_3 as $t \to \infty$ is of exponential type. For the corresponding solutions of (3.10), $\lim_{t\to\infty} \xi_3(t) = 0$ and $\lim_{t\to\infty} y(t) = \alpha$ imply $\lim_{t\to\infty} y_3(t) = \frac{\alpha}{1+a_3}$ so, for the corresponding solutions of (3.4), one obtains $\lim_{t\to\infty} y_1(t) = 0$, $\lim_{t\to\infty} y_2(t) = 0$, $\lim_{t\to\infty} y_3(t) = \frac{\alpha}{1+a_3}$, $\lim_{t\to\infty} y_4(t) = \alpha - \alpha\frac{a_3}{1+a_3} = \frac{\alpha}{1+a_3}$. We conclude that the equilibria of (3.2) are stable and that, by (3.3),

$$\begin{aligned}&\lim_{t\to\infty} z(t) = x, \quad \lim_{t\to\infty} v(t) = 0, \quad \lim_{t\to\infty} p_1(t) = \tilde{p}_2 + \frac{k}{S}x + \frac{\alpha}{1+a_3},\\ &\lim_{t\to\infty} p_2(t) = \tilde{p}_2 + \frac{\alpha}{1+a_3},\end{aligned} \tag{3.14}$$

if (z, v, p_1, p_2) is a solution starting from a small neighbourhood of the equilibrium $\left(x, 0, \tilde{p}_2 + \frac{k}{S}x, \tilde{p}_2\right)$.

For $\sigma < 0$ we follow the same steps. From (2.11) we get the system

$$\begin{cases} z_1 = v \\ \dot{v} = -\frac{k}{m}z - \frac{f}{m}v + \frac{S}{m}p_1 - \frac{S}{m}p_2 \\ \dot{p}_1 = \frac{C(x-z)}{V_0+Sz}\sqrt{p_1} - \frac{BS}{V_0+Sz}v \\ \dot{p}_2 = -\frac{C(x-z)}{V_0-Sz}\sqrt{p_a - p_2} + \frac{BS}{V_0-Sz}v. \end{cases} \tag{3.15}$$

It has the same equilibrium points as (3.1). Equation (3.3) gives

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\dfrac{k}{m}y_1 - \dfrac{f}{m}y_2 + \dfrac{S}{m}y_3 - \dfrac{S}{m}y_4 \\ \dot{y}_3 = -\dfrac{Cy_1}{V_0 + Sx + Sy_1}\sqrt{y_3 + \tilde{p}_2 + \dfrac{k}{S}x} - \dfrac{BSy_2}{V_0 + Sx + Sy_1} \\ \dot{y}_4 = \dfrac{Cy_1}{V_0 - Sx - Sy_1}\sqrt{p_a - \tilde{p}_2 - y_4} + \dfrac{BSy_2}{V_0 - Sx - Sy_1}. \end{cases} \tag{3.16}$$

The matrix of the linear part around zero is

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{f}{m} & \frac{S}{m} & -\frac{S}{m} \\ -\frac{C\sqrt{\tilde{p}_2 + \frac{k}{S}x}}{V_0+Sx} & -\frac{BS}{V_0+Sx} & 0 & 0 \\ \frac{C\sqrt{p_a - \tilde{p}_2}}{V_0-Sx} & \frac{BS}{V_0-Sx} & 0 & 0 \end{pmatrix} \tag{3.17}$$

and its characteristic polynomial is $Q_2(\lambda) = \lambda P_2(\lambda)$,

$$P_2(\lambda) = \lambda^3 + \frac{f}{m}\lambda^2 + \frac{\lambda}{m}\left(\frac{2BS^2V_0}{V_0^2 - S^2x^2} + k\right) + \frac{SC}{m}\left(\frac{\sqrt{\tilde{p}_2 + \frac{k}{S}x}}{V_0 + Sx} + \frac{\sqrt{p_a - \tilde{p}_2}}{V_0 - Sx}\right).$$

From the same Routh-Hurwitz criterion, P_2 will be stable if and only if

$$\frac{f}{m}\left(\frac{2BS^2V_0}{V_0^2 - S^2x^2} + k\right) - SC\left(\frac{\sqrt{\tilde{p}_2 + \frac{k}{S}x}}{V_0 + Sx} + \frac{\sqrt{p_a - \tilde{p}_2}}{V_0 - Sx}\right) > 0. \tag{3.18}$$

If inequality (3.18) is satisfied, the stability analysis proceeds as in the case $\sigma > 0$. Consider A_2 instead of A_1 in (3.7) and introduce η as in (3.8), then $\dot{\eta} = 0$ gives

$$a_1 = \frac{BS}{V_0 - Sx}\left(\frac{\sqrt{p_a - \tilde{p}_2}}{\sqrt{\tilde{p}_2 + \frac{k}{S}x}} - 1\right), \; a_2 = 0, \; a_3 = \frac{V_0 + Sx}{V_0 - Sx}\frac{\sqrt{p_a - \tilde{p}_2}}{\sqrt{\tilde{p}_2 + \frac{k}{S}x}}, \; a_4 = 1.$$

It is easy to see that, in the system that replaces (3.11), conditions (3.13) are satisfied. The Lyapunov-Malkin Theorem ensures, as before, the Lyapunov stability of equilibria in (3.15) and the same asymptotic behaviour.

4. Numerical results and simulations

We take the following values for the system parameters: $c_{\mathrm{d}} = 0{\cdot}6$, $f = 3 \times 10^3$ Ns/m, $k = 10^5$ N/m, $S = 10^{-3}$ m^2, $p_{\mathrm{a}} = 2 \times 10^7$ N/m^2, $B = 6 \times 10^8$ N/m^2, $\rho = 850$ kg/m^3,

Table 1 The coordinates of seven points.

	1	2	3	4	5	6	7
x [m]	−0·025	−0·015	0·025	−0·025	0·025	−0·001	−0·0015
p	0·95	0·5	0·05	0·05	0·95	0·75	0·05

$V_0 = 3 \times 10^{-5}$ m^3, $\lambda = 2/3$, $w = 8{\cdot}5 \times 10^{-4}$ m. We have the limitations $|x| < 3 \times 10^{-2}$ m $:= x_{\mathrm{M}}$ and $|\sigma| < 10^{-3} := \sigma_{\mathrm{M}}$.

Introduce $p \in (0, 1)$ by

$$p = \frac{\tilde{p}_2}{p_a}. \tag{4.1}$$

Then the stability borders are given by

$$\begin{aligned} g_1(x,p) \;:=\; & 2fBS^2V_0 + fk\left(V_0^2 - S^2x^2\right) - \\ & - mSC\sqrt{p_a}\left[(V_0 - Sx)\sqrt{1-5x-p} + (V_0 + Sx)\sqrt{p}\right] = 0, \end{aligned} \tag{4.2}$$

$$g_1 \;:\; (-x_M, x_M) \times (0,1) - \left\{(x,p)\middle|\, 1 - 5x - p \le 0\right\} \to \mathrm{R}, \text{ for } \sigma > 0$$

and

$$\begin{aligned} g_2(x,p) \;:=\; & 2fBS^2V_0 + fk\left(V_0^2 - S^2x^2\right) - \\ & - mSC\sqrt{p_a}\left[(V_0 - Sx)\sqrt{5x+p} + (V_0 + Sx)\sqrt{1-p}\right] = 0, \end{aligned} \tag{4.3}$$

$$g_2 \;:\; (-x_M, x_M) \times (0,1) - \left\{(x,p)\middle|\, 5x + p \le 0\right\} \to \mathrm{R}, \text{ for } \sigma < 0$$

For $m = 30$ kg, we have $g_1(x, p) > 0$, $g_2(x, p) > 0\ \forall\ (x, p)$ in their domains, so we conclude that the equilibria (3.2) are stable and (3.14) holds, too. This situation is presented in Figure 2 and, in Figure 3, for $m = 60$ kg, and $x = -0{\cdot}0015$ m.

By use of a root-locus-type approach, the stability borders for $m = 60$ kg are depicted in Figure 4. A quasi-mirror symmetry of the drawing in the pair of cases $\sigma > 0$ and $\sigma < 0$ can be observed. The thick lines mark the definition domains of the $g_1(x, p)$, $g_2(x, p)$ maps. The seven points described in Figure 4a, were represented on (x, p) plan, the sign '+' showing the stable equilibrium point, the sign '−' showing the unstable equilibrium point and the indicator 'c' marks the fact that the point gives a complex value to the $g(x, p)$ function.

Starting with the initial conditions $z(0) = x - \sigma_0/\lambda$, $v(0) = 0$, $p_1(0) = p_2(0) + 5z(0)$, $p_2(0) = qpp_{\mathrm{a}}$, $q \in (0, 1/p)$, $\sigma_0 \in (-\sigma_{\mathrm{M}}, \sigma_{\mathrm{M}})$, we integrate numerically the system (3.1) as long as $\sigma > 0$, turn to (3.15) if $\sigma < 0$, coming back to (3.1) when $\sigma > 0$ and so on.

Basically, confirmation of theoretical predictions by model simulations can be an impossible and even unnecessary task. Owing to the intrinsic switching structure (2.9–2.10) of our model, an extra difficulty appears: at each integration step, taking into account the sign induced to the relative displacement σ, we observe that the values of the state variables considered as initial conditions influence the system stability. So, a rigorous prediction concerning

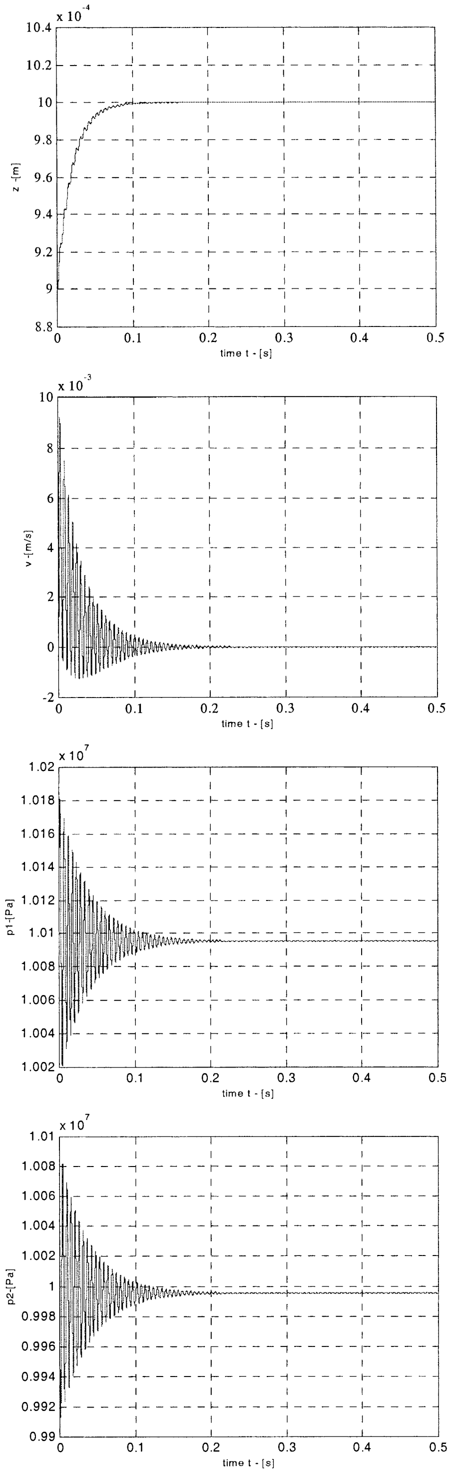


Figure 2. Numerical results for the servomechanism with $m = 30$ kg, $x = 0{\cdot}001$ m, $z_0 = 0{\cdot}0009$ m, $V_0 = 0$ m/s, $p_{10} = 101 \times 10^5$ N/m^2, $p_{20} = 101 \times 10^5$ N/m^2.

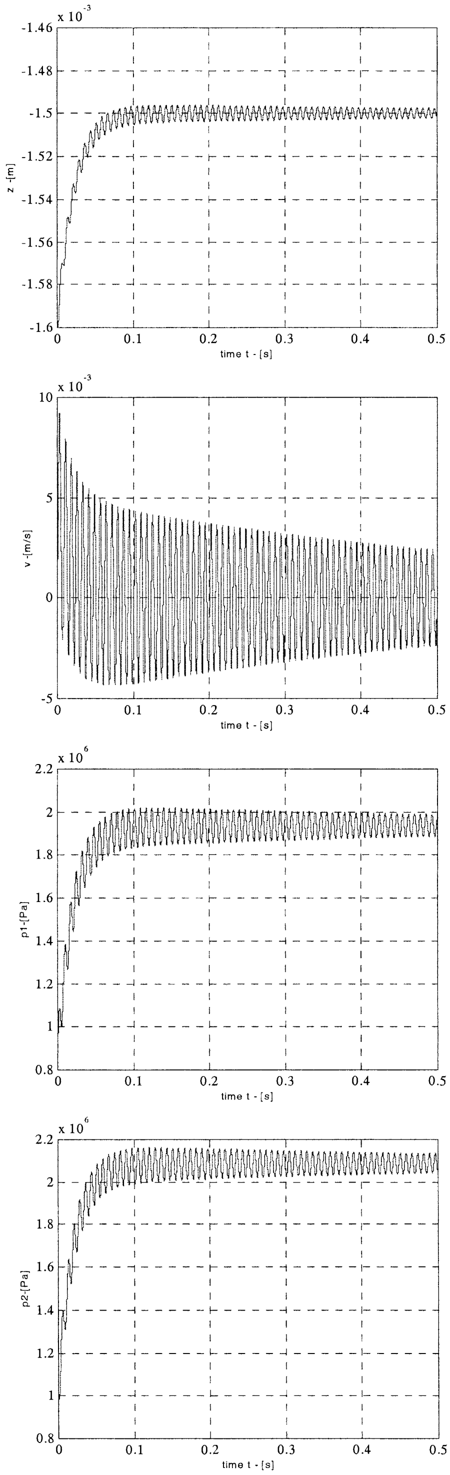
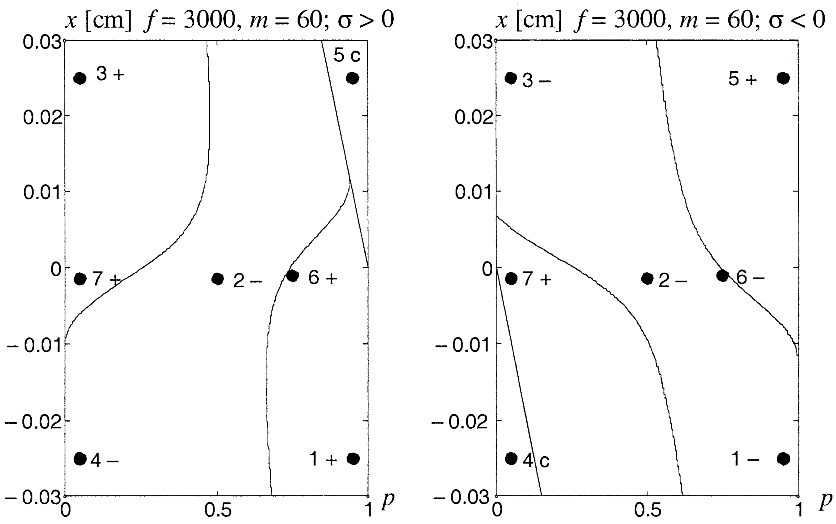


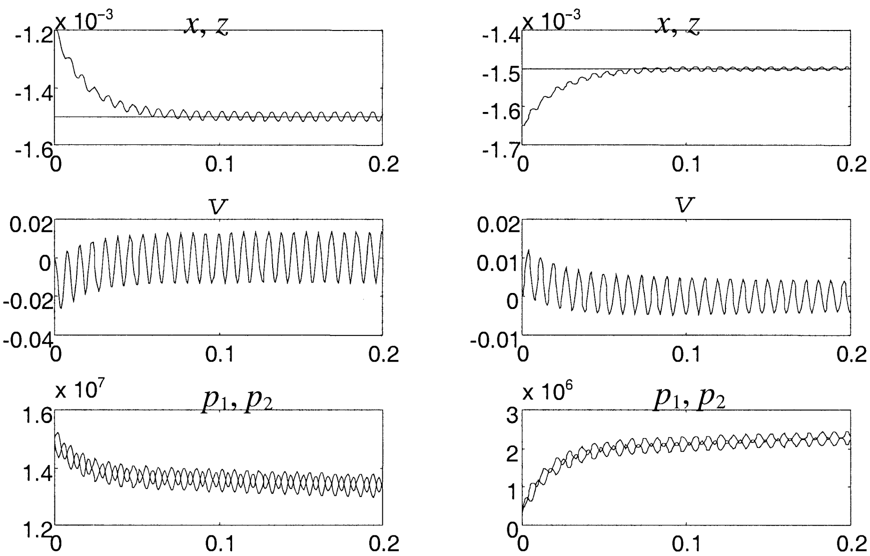
Figure 3. Numerical results for the servomechanism with $m = 60$ kg, $x = -0{\cdot}0015$ m, $z_0 = -0{\cdot}0016$ m, $V_0 = 0$ m/s, $p_{10} = 100 \times 10^5$ N/m^2, $p_{20} = p_{10} + 0{\cdot}005$ N/m^2.

(a)



(b)

Figure 4. Stability borders of the four-dimensional nonlinear model of hydraulic servomechanism.



a) b)

Figure 5. Time history of state variables (time in seconds).

system stability or instability of the theoretical hydraulic-servomechanism model is difficult to be ascertained in parameter configurations such as those represented in Figure 4, the case $m = 60$ kg, in which domains of stability or instability coexist.

To illustrate these difficult features, some illustrative time histories of state variables are herein retained in Figure 5. Initial conditions relative to equilibrium point 2, with $\sigma < 0$, $\sigma_0 = -\sigma_M/5$ and $q = 1{\cdot}5$, in the case of Figure 5a and to equilibrium point 7, with $\sigma > 0$, $\sigma_0 = \sigma_M/10$ and $q = 0{\cdot}5$, in the case of Figure 5b are used. The first case illustrates a situation of instability and the second a situation of stability. The pictures remain almost unchanged when the degree of accuracy is increased. One can easily see the effect of switching in the phase portrait in Figure 5b, when compared with Figure 3.

Problems of the kind investigated in this paper frequently arise in the aviation industry. Given the aviation regulations, only the fulfilment of the stability conditions *in the whole domain* where (x, p) are defined, can be accepted as a preliminary condition in the synthesis of a hydraulic servomechanism.

The existence of arbitrary initial pressures that satisfy $0 < p_i(0) < p_a$, $i = 1, 2$ in various dynamic processes in hydraulic servomechanisms has been verified by laboratory measurements (see [29]). Recall that, according to [30], if one is given a value m for which stability of equilibria is ascertained, one must have laboratory stability tests also for $2m$. Some results of such tests are to be found in [21], leading to the conclusion that dynamical stability is observed also in the case $m = 60$ kg.

5. Concluding remarks

A natural setting for the investigation of various properties of a mechano-hydraulic servomechanism is a four-dimensional nonlinear system of ordinary differential equations.

Although not new, the detailed derivation of such a model, starting from physical principles that govern the dynamics of the spool valve, can be considered as a contribution to the dissemination of more accurate approaches to this class of servomechanisms.

The inherent complexity of the nonlinearities of valve-controlled hydraulic servomechanisms yields difficulties in finding analytical conditions that ensure the stability of equilibrium points. The main contribution of this work is the statement of such conditions. The classical Routh-Hurwitz criterion is combined with a theorem of Lyapunov and Malkin to handle the critical case due to the presence of a zero eigenvalue in the spectrum of the Jacobian matrix calculated at equilibria. As a result inequalities (3.6) and (3.18) are proved to be sufficient for the Lyapunov stability of equilibria. We underline that these conditions must be satisfied for all the values (x, p) that parameterize the equilibria. The designer faced with the synthesis of a mechano-hydraulic servomechanism can regard these conditions as a reference point in the design.

The switching-type structure of the mathematical model leads to two inequalities (already mentioned) that ensure stability and accordingly to stability borders. The investigation of the stability of equilibria on the boundary could be accomplished along the lines as indicated in [31].

The same switching structure is also considered when the dynamics of the system is simulated numerically. The resulting phase pictures show the influence of switching on the behavior of solutions. Similar results work for electrohydraulic servomechanisms where instead of $\sigma = \lambda(x - z)$ one considers, for instance, $\sigma = k_{\text{em}}(u - k_p z)$ with k_{em} as electro-mechanical conversion gain, k_p the displacement transducer gain and u the reference signal [V] (see [10, 15]). To avoid instability due to violation of the inequalities (3.6) or (3.18) or to the switching structure, one can use a stabilizing controller. This idea will be developed in forthcoming papers.

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